AVERAGING SCHEMES FOR INHOMOGENEOUS UNIVERSE AND TIMESCAPE COSMOLOGIES

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Abstract

Timescape model is a phenomenological cosmology model without dark energy. By revisiting Einstein's strong equivalence principle and extending at the general average of the cosmological Einstein's equation, one can construct averaging schemes and timescape model. Detailed description of that timescape cosmology and expansion laws in general relativity within a covariant fluid approach have been studied. Physical interpretation of the results are carried out and visualizations of interesting results are implemented.

Keywords: Timescape model, dark energy, averaging schemes.

Introduction

Dark energy has been described as the biggest problem in cosmology. In a different perspective, dark energy is not the internal energy of a mysterious fluid, but a misidentification of those aspects of cosmological gravitational energy which by virtue of the equivalence principle cannot be localized: gradients in the energy associated with the varying curvature of space and the varying kinetic energy of the expansion of space. These are important aspects of gravitational physics in universe, which at the present epoch is very inhomogeneous, dominated by voids. (Adler. R, Bazin.M, Schiffer. M, 1975)

One uses the formalism of Thomas Buchert in taking account of backreaction in the evolution of Einstein's equations. One crucial insight is that gravitational energy and clock rates are defined with respect to a notion of infinity. Bound system where space is not expanding, including all galaxies live within finite infinity, but volume average positions in free expanding space lie beyond it there is a difference in gravitational energy and spatial curvature between the two locations. The differences were initially miniscule but are large today. Taking account of the initial conditions set by primordial inflation at the time of last scattering, when the cosmic microwave background was laid down, a quantitative model of the universe is developed. Relative to bound system observers, ideal observers at volume average positions in voids will measure an older age of the universe, a lower isotropy. These differences can be systematically quantified. On account of the variance in clock rates volume average observers in voids infer no apparent "cosmic acceleration", but observers in bound systems do. Apparent acceleration begins when the void volume fraction reaches 59%, at a redshift of order z = 0.5 to 1.0 (depending on whether one uses the CMB or supernovae as an estimator). The mystery of dark energy is explained purely in Einstein's theory, through a deeper understanding of those parts of general relativity, which Einstein himself recognised as being difficult: the understanding of gravitational energy, given that space itself is dynamical and may contain energy and momentum. (Buchert.T, 1999)

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Although matter in the Universe was extremely uniform when the cosmic microwave background radiation formed, since then gravitational instability led to an increasingly complex hierarchy of structures at late epochs a cosmic web of voids with galaxies and clusters in sheets, filaments and knots. In the standard model of cosmology this hierarchy is treated by assuming an average background universe which expands uniformly, just as if no structures were present. The nonlinear growth of structure is then formally treated and simulates using Newtonian gravity only. In other words, the background evolution and the evolution of structure do not couple to each other, an idealization at odds with the very foundations of general relativity which generically demand a coupling between matter and geometry.

After that, Friedmann–Lemaître–Robertson–Walker (FLRW) model keeps spatial curvature uniform everywhere and decouples its evolution from that of matter, which is again not a generic consequence of Einstein's equations. The difference between an averaged generic evolution and an ideal FLRW evolution is usually called backreaction, and is potentially significant for interpreting observations of the actual inhomogeneous Universe. (Buchert.T, 1999)

Dark Energy and Dark Matter

In the standard model of cosmology one has to conjecture the existence of two constituents, if observational constraints are met, that both have unknown origin: first, a dominant repulsive component is thought to exist that can be modeled either by a positive cosmological constant or a scalar field, e.g. a so-called quintessence field. Besides this Dark Energy, there is, secondly, a non-baryonic component that should considerably exceed the contribution by luminous and dark baryons and massive neutrinos. The Dark Matter is thought to be provided by exotic forms of matter, not detected in (non-gravitational) experiments. According to the concordance model, the former converges to about ¾ and the latter to about ¼ of the total source of Friedmann's equations, up to a few percent that have to be attributed to baryonic matter and neutrinos (in the matter-dominated era).(Buchert.T, 2007)

Averaging Strategies: Different 'Directions' of Backreaction

The notion of averaging in cosmology is tied to space-plus-time thinking. Despite the success of general covariance in the four-dimensional formulation of classical relativity, the cosmologist's way of conceiving the Universe is evolutionary. This breaking of general covariance is in itself an obstacle to appreciating the proper status of cosmological equations. The standard model of cosmology is employed with the implicit understanding that there is a global spatial frame of reference that, if mapped to the highly isotropic Cosmic Microwave Background, is elevated to a physical frame rather a particular choice of a mathematical slicing of spacetime. (Witshire. D. L, 2007)

This point is raised as a criticism of an averaging framework, as if this problem were not in the standard model of cosmology. Again, the 'natural' choice for the matter model 'irrotational dust' is a collection of freely-falling continuum elements, now for an inhomogeneous continuum. For such a generalized collection of fundamental observers, the 4-metric form reads

$$^{4}g = -dt^2 + \, ^{3}g \, ; \;\; ^{3}g = g_{ab} \, dX^a \otimes dX^b \, ,$$

where latin indices run through 1…3 and X^a are local (Gaussian normal) coordinates. Evolving the first fundamental form ${}^3g$ of the spatial hypersurfaces along $\partial/\partial t =: \partial_t$ defines their second fundamental form

$$ {}^3K = K_{ab} dX^a \otimes dX^b \; ; \; K_{ab} := -\frac{1}{2}\,\partial_t\, g_{ab} \,, $$

with the extrinsic curvature components K_{ab}. Such a comoving (synchronous) slicing of spacetime may be considered 'natural', but it may also be questioned.[(Buchert.T, Ehlers.J, 1995]

Effective Description of Inhomogeneous Universe Models

Restricting attention to a universe filled with irrotational dust, i.e. irrotational pressureless matter, one spatially average the scalar parts of Einstein equations with respect to a collection of comoving (generalized fundamental) observers over a compact, rest mass preserving spatial domain D, and obtain the following set of equations

$$ \left(\frac{\dot{a}_D}{a_D}\right)^2 - \frac{8\pi G}{3}\langle e\rangle_D = -\frac{\langle R\rangle_D + Q_D}{6}, \tag{1} $$

$$ \frac{\ddot{a}_D}{a_D} + \frac{4\pi G}{3}\langle e\rangle_D = \frac{Q_D}{3}, \tag{2} $$

$$ \langle e\rangle^{\bullet}{}_D + 3\frac{\dot{a}_D}{a_D}\langle e\rangle_D = 0, \tag{3} $$

$$ \frac{1}{a_D^6}(Q_D a_D^6) + \frac{1}{a_D^2}(\langle R\rangle_D a_D^2)^{\bullet} = 0, \tag{4} $$

where a_D is the effective volume scale factor

$$ a_D(t) = \left(\frac{V_D(t)}{V_{D_i}}\right)^{1/3}, \tag{5} $$

with V_{D_i} the initial volume of the domain and $V_D(t)$ its volume at a proper time t, $\langle e\rangle_D = M a_D^{-3}/V_D$, the density of irrotational dust averaged over $D, \langle R\rangle_D$ the spatial scalar curvature averaged over D and Q_D the kinematical backreaction

$$ Q_D(t) := \frac{2}{3}\left\langle\left(\theta - \langle\theta\rangle_D\right)^2\right\rangle_D - 2\langle\sigma^2\rangle_D \,, \tag{6} $$

with θ the rate of expansion and $\sigma := \sqrt{\frac{1}{2}\sigma^{ij}\sigma_{ij}}$ the rate of shear with the shear tensor components σ_{ij}. Equations (1) and (2) govern the kinematics of the effective scale factor and equations (3) and (4) express the conservation law for the dust matter and the backreaction terms, respectively.

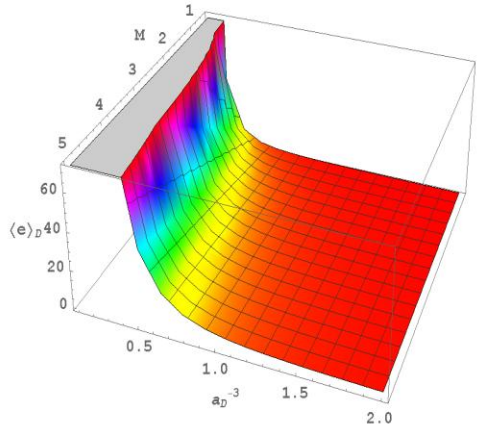
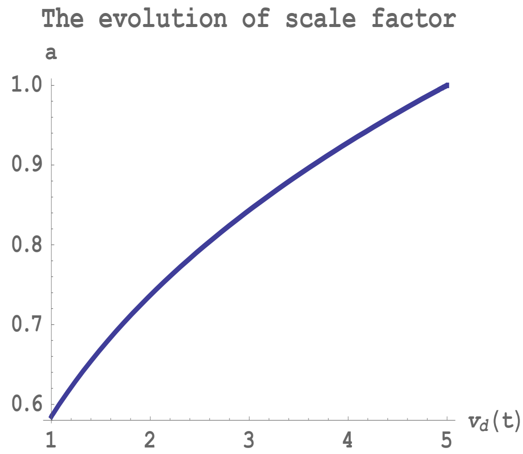


Figure 1 The evolution of density of irrotational dust in terms of mass and scale factor.



Figures 2 The evolution of scale factor with $V_D(t)$.

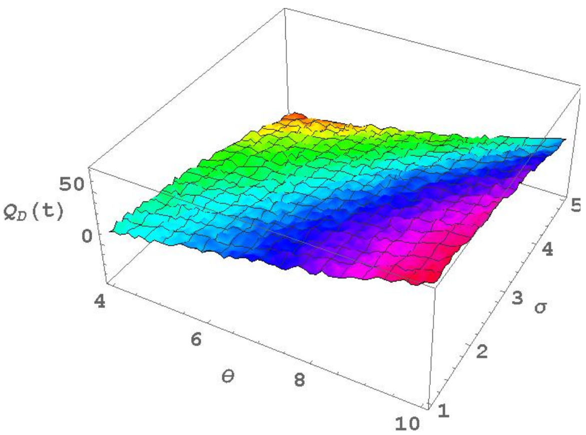


Figure 3 Variation of Kinematical Backreaction with expansion scalar and shear.

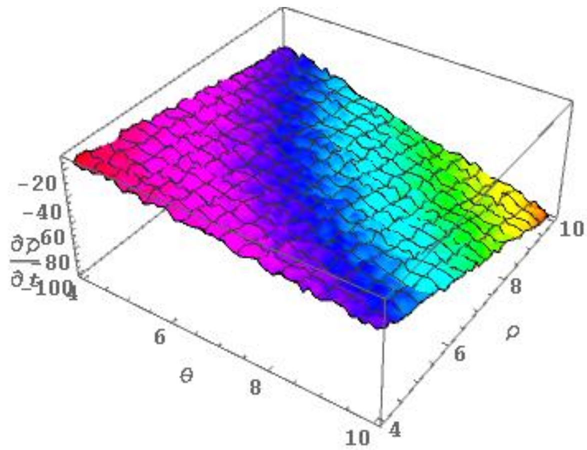


Figure 4 Time evolution of energy density in terms of scalar and density.

General Expansion Law in Newtonian Cosmology

Non Commutation Time Evolution and Averaging

Let us consider a portion of the Universe *D* (*t*) with volume *V*(*t*). Henceforth, one concentrates on the expansion which one describes by the local expansion scalar $\theta = \nabla.\vec{v}$. Introducing the scale factor via the volume, $a_D \coloneqq V^{1/3}$, one can write the spatial average of θ on the domain *D* as

$$\langle\theta\rangle_D = \frac{1}{V}\int_D d^3\,x\,\theta = \frac{\dot{V}}{V} = 3\,\frac{\dot{a}_D}{a_D} \qquad (7)$$

As written in (7) the spatial average can be calculated as a simple Euclidean volume integral over the domain *D*, the mail advantage of a Newtonian treatment. The subscript *D* indicates that the averages (as well as the scale factor) depend on morphological properties of the spatial such as content, shape and connectivity.

One evaluates the evolution of the average in a tube of trajectories of fluid elements, i.e., one introduces a Lagrangian mapping $\vec{f_t} : \vec{x} = \vec{f}\left(\vec{X}, t\right)$ which sends fluid particles from their initial (Lagrangian) position $\vec{X}$ to their final (Eulerian) position $\vec{x}$. One uses the Jacobian of this mapping, $d^3 = \mathrm{J}d^3X$, J: = det $\left(\frac{\partial f_i}{\partial X_k}\right)$, to transform spatial averages to volume integrals in Lagrangian space:

$$\langle\theta\rangle_D = \frac{1}{V}\int_{D(t)} d^3\, x\, \theta(\vec{x}, t) = \frac{1}{V}\int_{D(t_0)} d^3\, X\, \mathrm{J}\left(\vec{X}, t\right)\theta\left(\vec{X}, t\right). \tag{8}$$

Using the Lagrangian time-derivative, $\frac{d}{dt} := \frac{\partial}{\partial t} + \nabla.$, one obtains the nonlinear commutation rule. (Buchert. T, Ehlers. J, 1995)

$$\frac{d}{dt}\langle\theta\rangle_D - \langle\frac{d}{dt}\theta\rangle_D = \langle\theta^2\rangle_D - \langle\theta\rangle^2{}_D. \tag{9}$$

Equation (9) shows that the evolution of the average and the average over the evolved field do not commute, their difference being given by the nonlinear fluctuation term on the right hand side.

The Generalized Friedmann Equation

Averaging Raychaudhuri's equation for the evolution of the expansion scalar, the scale factor a_D is found to obey the general expansion law. (Carroll, S. M., 2004)

$$3\frac{\ddot{a}_D}{a_D} + 4\pi G\,\frac{M_D}{a_D^3} - \Lambda = -\mathbf{Q} \tag{10}$$

where the source term **Q** depends on the fluctuation term in (2.3) and the magnitudes of rotation (ω) and shear (σ) of the flow,

$$\mathbf{Q} := \frac{2}{3}\left(\langle\theta\rangle^2{}_D - \langle\theta^2\rangle_D\right) + 2\langle\sigma^2 - \omega^2\rangle_D \tag{11}$$

M_D denotes the total mass contained in D .

Eq. (11) may be rewritten as a standard Friedmann equation for the actual source term $\langle\varrho_{eff}\rangle_D$:

$$4\pi G\,\langle\varrho_{eff}\rangle_D \coloneqq 4\pi G\langle\varrho\rangle_D + \mathbf{Q}, \tag{12}$$

where $\langle\varrho\rangle_D = M_D/a_D^3$ is the pure average matter density. Eq. (12) shows that, for irrotational flows ,the additional "dynamical mass" is a positive term which adds to the matter density, if σ^2 is larger than the fluctuation $\langle\theta\rangle^2{}_D - \langle\theta^2\rangle_D = \langle(\theta - \langle\theta\rangle_D)^2\rangle_D \geq 0$. This suggests to add the source term **Q** to the list of dark matter candidates: strongly sheared inhomogeneities could "fake" an additional density which leads to an overestimate of the density parameter.

Integrating eq.(11) with respect to time yields the generalized form of Friedmann's differential equation:

$$\frac{\dot{a}_D^2 + k}{a_D^2} - \frac{8\pi G M_D}{3a_D^3} - \frac{\Lambda}{3} = \frac{1}{3a_D^2}\int_{t_0}^{t} dt'\,\mathbf{Q}\frac{d}{d\acute{t}}\,a_D^2. \tag{13}$$

Averaging Globally Homogeneous (Isotropic) Universes

One now assumes that a global Hubble flow exists on some large scale A and that the expansion-factor on that scale obeys Friedmann's differential equation Eq.(13) for (**Q = 0**);

on the scale A we write $a_D \equiv a$. Splitting the velocity gradient $v_{i,j}$ into its Hubble part and a peculiar-velocity gradient $v_{i,j} = H(t)\,\delta_{i,j} + v_{i,j}$, where H(t) = $\frac{\dot{a}}{a}$, one obtains:

$$\theta = 3H + \nabla.\vec{u} \qquad (14)$$

After averaging, the last equation leads to a relation Between the Hubble function H(t), the "effective Hubble function" $H_D(t) := \frac{\dot{a}_D}{\dot{a}_D}$, and the peculiar-velocity field $\vec{u}(\vec{x},t)$:

$$H(t) = H_D(t) - \frac{1}{3} a_D^{-3} \int_{\partial D(t)} \vec{d}\,S.\vec{u}\,. \qquad (15)$$

H_D may be interpreted as that Hubble function which is inferred from the (possibly anisotropic and rotational) dynamics of the spatial domain D. (This interpretation is possible if statistical averages of many such spatial domains are considered, but at present, one only measures one member of such an ensemble.) Accordingly, the source term **Q** can be split into its Hubble part and deviations thereof and transformed into surface integrals over the boundary $\partial D(t)$. (Rosanen. S, 2006)

$$3\frac{\ddot{a}_D}{a_D} + 4\pi G\frac{M_D}{a_D^3} - \Lambda = -\frac{2}{3}\left(a_D^{-3}\int_{\partial D}\vec{d}\,S.\vec{u}\right)^2 + a_D^{-3}\int_{\partial D}\vec{d}S.(\vec{u}\,\nabla.\vec{u} - \vec{u}.\nabla\vec{u})\ . \qquad (16)$$

Averaging and Schemes

The idea that the large scale universe is homogeneous and isotropic necessarily entails an implicit notion of averaging on these large scales. If g denotes the metric, Γ the Christoffel connection and E[g] the Einstein tensor for the metric g, the one has the relations

$$\Gamma \sim \partial g\ ;\ E[\,g\,] \sim \partial\Gamma + \Gamma^2\ , \qquad (17)$$

with ∂ denoting spacetime derivatives. The Einstein equations are therefore

$$E[\,g\,] = T, \qquad (18)$$

with T denoting the energy-momentum tensor of the matter components. Now, irrespective of any details of the averaging operation, one notes that

$$E[< g >] - < E[g] > \sim < \Gamma >^2 - < \Gamma^2 > \neq 0\ , \qquad (19)$$

with the angular brackets denoting the averaging. The FLRW solution would amount to solving the equations $E[< g >] = < T >$. In general, therefore it is not true that averaging out the fluctuating inhomogeneities leaves being the FLRW solution, since what one is actually left with is

$$E[< g >] = < T > - C\ ;\ C \sim < \Gamma^2 > - < \Gamma >^2\ , \qquad (20)$$

and the homogeneous solution that we are looking for will depend on the details of the correction terms C. The second cause for concern comes from observations. It has now been established beyond a reasonable doubt, that the FLRW metric confronted with observations indicates an accelerating scale factor. (Witshire. D. L, 2007)

Buchert's Spatial Averaging

The most straightforward and intuitively clear application of Buchert's spatial averaging is in the case when the matter source is a pressure "dust" with an energy-momentum tensor

$T^{ab} = \rho u^a \ u^b$, with u^a the dust 4-velocity which satisfies $u_a u^a = -1$. Assuming further that the dust is irrotational, the 4-velocity will be orthogonal to 3-dimensional spatial sections and the metric can be written in "synchronous and comoving" coordinates (in which $u^a = 1, \vec{0}$)as

$$ds^2 = -dt^2 + h_{AB}(t, \vec{x}) dx^A \ \ dx^B \qquad (21)$$

The expansion tensor Θ^A_B is given by $\Theta^A_B = (1/2) h^{AC} \dot{h}_{CB}$ where the dot refers to a derivative with respect to time *t*. The traceless symmetric shear tensor is defined as $\sigma^A_B \ \equiv \Theta^A_B - (\Theta/3)\ \delta^A_B$ where $\Theta = \Theta^A_A$ is the expansion scalar.

The scalar equations are the Hamiltonian constraint (3.6a) and the evolution equation for Θ

$$^{(3)}R + \frac{2}{3}\Theta^2 - 2\sigma^2 = 16\pi G\rho\ , \qquad (22)$$

$$^{(3)}R + \dot{\Theta} + \Theta^2 = 16\pi G\rho\ , \qquad (23)$$

where $^{(3)}R$ is the Ricci scalar of the 3-dimensional hypersurface of constant *t* and σ^2 is the rate of shear defined by $\sigma^2 \equiv (1/2)\sigma^A_B \sigma^B_A$. Eqns. (22) and (23) can be combined to give Raychaudhuri's equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 + 4\pi G\rho = 0\ . \qquad (24)$$

The continuity equation $\dot{\rho} = -\Theta\rho$ which gives the evolution of ρ, is consistent with Eqns. (23) , (24). One only consider the scalar equations, since the spatial average of a scalar quantity can be defined in a gauge covariant manner within a given foliation of spacetime. For the spacetime described by (24), the spatial average of a scalar $\psi(t, \vec{x})$ over a comoving domain $\mathcal{D}$ at time *t* is defined by

$$< \Psi >_{\mathcal{D}} = \frac{1}{V_D} \int_D d^3x \sqrt{h}\, \psi\ , \qquad (25)$$

where *h* is the determinant of the 3-metric h_{AB} and V_D is the volume of the comoving domain given by $V_D = \int_D d^3x \sqrt{h}$. The following commutation relation then holds

$$< \Psi >_{\mathcal{D}} - < \dot{\Psi} >_{\mathcal{D}} = \ < \Psi\Theta >_{\mathcal{D}} - < \Psi >_{\mathcal{D}} < \Theta >_{\mathcal{D}}\ , \qquad (26)$$

which yields for the expansion scalar Θ

$$< \Psi >_{\mathcal{D}} - < \dot{\Theta} >_{\mathcal{D}} = \ < \theta^2 >_D - < \Theta >^2_D\ . \qquad (27)$$

Introducing the dimensionless scale factor $a_D \equiv (V_D / V_{Din})^{1/3}$ normalized by the volume of the domain $\mathcal{D}$ at some initial time t_{in} , we can average the scalar Einstein equations (22), (23) and the continuity equation to obtain

$$\left(\frac{\dot{a_D}}{a_D}\right)^2 = \frac{8\pi G}{3} < \rho >_D - \frac{1}{6}(\mathcal{Q}_D + < R >_D), \qquad (28)$$

$$\left(\frac{\ddot{a_D}}{a_D}\right) = -\frac{4\pi G}{3} < \rho >_D + \frac{1}{3}\ \mathcal{Q}_D\ , \qquad (29)$$

$$< \rho >_D = - < \Theta >_{\mathcal{D}} < \rho >_D = \ -3 \frac{\dot{a_D}}{a_D} \ < \rho >_D\ . \qquad (30)$$

Concluding Remarks

It is concluded that timescaping nature of cosmology can be more smoothly applicable to fundamental cosmological entities such as scale factor and density parameters and extendable to different time evolution settings. Some interesting visualization of the results are implemented using Mathematica software.

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